Linear Algebra, Spring 2005

Solutions

May 4, 2005

Problem 4.104

4.104 a

$$\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & -1 & -3 & 2 & -1 \\ 0 & -1 & -4 & 1 & 1 \\ 0 & -1 & -1 & 4 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 2 & 2 & -4 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank is 3 [three non-zero rows in the echelon matrix.]

4.104 b

$$\begin{bmatrix} 1 & 2 & -3 & -2 \\ 1 & 3 & -2 & 0 \\ 3 & 8 & -7 & -2 \\ 2 & 1 & -9 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -2 & -4 \\ 0 & 3 & 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank is 2.

4.104 c

$$\begin{bmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 5 & 8 & 1 \\ -1 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & -3 & 9 \\ 0 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank is 3.

Problem 4.105

(a)

For n_1 we are looking for subsets of column space of the matrix which have one member and they are linearly independent. Any subset with one member (non-zero) is linearly independent. A subset including zero vector is always linearly dependent. Therefore, for n_1, the following subsets are linearly independent: $\{c_1\}$, $\{c_2\}$, $\{c_4\}$, $\{c_5\}$. Then $n_1 = 4$. (note: c_i is the ith column of the matrix.)

For n_2 we are looking for subsets of the form $\{c_i, c_j\}$ which are linearly independent. c_3 must be excluded and $\{c_1, c_4\}$ is linearly dependent, other two pairs vectors are linearly independent. For a two-member subset $\{c_i, c_j\}$ if $c_i = kc_j$ the subset is linearly dependent, otherwise the subset is linearly independent. Generally if in any subset there is two vector that $c_i = kc_j$, that subset is linearly dependent. $\{c_1, c_2\}$, $\{c_1, c_5\}$,$\{c_2, c_4\}$,$\{c_2, c_5\}$,$\{c_4, c_5\}$ are linearly independent. Therefore $n_2 = 5$.

For n_3 we have to exclude c_3 and one of the two columns c_1 or c_4. We examine $\{c_1, c_2, c_5\}$ by converting to echelon form we will see that these three vectors are linearly dependent. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ So $n_3 = 0$.

n_4 and n_5 are zero because we have to include the Null column or one of the two columns c_1 or c_4.

(b)

For n_1, $\{c_1\}$, $\{c_2\}$, $\{c_3\}$, $\{c_5\}$ are linearly independent, therefore, $n_1 = 4$.

For n_2, $\{c_1, c_2\}$, $\{c_1, c_3\}$, $\{c_1, c_5\}$, $\{c_2, c_3\}$, $\{c_2, c_5\}$, $\{c_3, c_5\}$ are linearly independent. Therefore, $n_2 = 6$.

For n_3, we examine $\{c_1, c_2, c_3\}$, $\{c_1, c_2, c_5\}$, $\{c_1, c_3, c_5\}$, $\{c_2, c_3, c_5\}$ by converting to echelon form. The echelon forms show all of them are linearly independent. $n_3 = 4$.

$n_4, n_5 = 0$ because in R^3 and set with more than three vector are linearly dependent.

note: textbook answers for n_2, n_3 are not correct.

Problem 4.106

(a-i)

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 6 \\ 2 & 4 & 3 & 8 & 3 & 9 \\ 1 & 2 & 2 & 5 & 3 & 11 \\ 4 & 8 & 6 & 16 & 7 & 26 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 6 \\ 0 & 0 & 1 & 2 & 1 & -3 \\ 0 & 0 & 1 & 2 & 2 & 5 \\ 0 & 0 & 2 & 4 & 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 6 \\ 0 & 0 & 1 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 1 & 8 \end{bmatrix} \rightarrow$$

$$\text{(echelon form)} \begin{bmatrix} (\mathbf{1}) & 2 & 1 & 3 & 1 & 6 \\ 0 & 0 & (\mathbf{1}) & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & (\mathbf{1}) & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$ ($-r_2$ is added to r_1, see *row*

canonical form on page 74 of the textbook) $\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\rightarrow$ (*row canonical form*) $\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 2 & 0 & -11 \\ 0 & 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(a-ii)

By looking at echelon form: C_2, C_4, C_6 (see page 132 of the textbook)

(a-iii)

C_1, C_3, C_5 can not be written as a linear combination of the other columns, therefore they establish a basis for the column space.

(a-iv)

$C_6 = xC_1 + yC_3 + zC_5$

$$\begin{cases} x + y + z &= 6 \\ 2x + 3y + 3z &= 9 \\ x + 2y + 3z &= 11 \\ 4x + 6y + 7z &= 26 \end{cases}$$

to solve this set of equations we convert following matrix to the row canonical form:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 3 & 3 & 9 \\ 1 & 2 & 3 & 11 \\ 4 & 6 & 7 & 26 \end{bmatrix} \to \text{(row canonical form)} \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$x = 9, y = -11, z = 8$

An alternative solution: Consider the row-canonical form of the matrix. Form a linear combination and equate to zero: $xC_1 + 0C_2 + yC_3 + 0C_4 + zC_5 - C_6 = 0$. We can immediately see that the requires: $x = 9, y = -11, z = 8$. So we can read linear combinations that make up the dependent rows directly off the row-canonical form of the original matrix without doing any further work.

(b-i)

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 2 & 1 \\ 2 & 4 & 5 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 4 & 6 \\ 3 & 6 & 7 & 7 & 9 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 3 & 2 & 5 \\ 0 & 0 & 1 & 4 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 & 2 & 4 \end{bmatrix} \to$$

$$\text{(echelon form)} \begin{bmatrix} (\mathbf{1}) & 2 & 2 & 1 & 2 & 1 \\ 0 & 0 & (\mathbf{1}) & 2 & 1 & 3 \\ 0 & 0 & 0 & (\mathbf{1}) & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -3 & 0 & -5 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\text{(row canonical form)} \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b-ii)

We can infer from echelon form that C_2, C_5, C_6 are linear combinations of the preceding columns. (see page 132 of textbook)

(b-iii)

From echelon form: C_1, C_3, C_4

(b-iv)

$C_6 = xC_1 + yC_3 + zC_4$

$$\begin{cases} x + 2y + z & = 1 \\ 2x + 5y + 4z & = 5 \\ x + 3y + 4z & = 6 \\ 3x + 7y + 7z & = 10 \end{cases}$$

Following matrix represents above set of equations: $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 5 & 4 & 5 \\ 1 & 3 & 4 & 6 \\ 3 & 7 & 7 & 10 \end{bmatrix}$

To solve the equations, we convert the matrix to the row canonical form:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow x = 1, y = -1, z = 2$$

As before we can read this solution directly off the corresponding column in the row-canonical form of the original matrix.

Problem 4.107

To see if the row spaces are the same we find the space spanned by the rows of each matrix after reducing to echelon form and compare.

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 3 & -4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 \\ 0 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & -5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -1 & 10 \\ 3 & -5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 0 & -1 & -4 \\ 0 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Looking at the matrices, A and C have the same row space.

Problem 4.108

Form the matrices A, B, C corresponding to U_1, U_2 and U_3 and row reduce these matrices to canonical form

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 & -3 \\ 3 & -2 & -8 \\ 2 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -3 \\ 0 & -1 & -1 \\ 0 & -3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 3 & -1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

U_1 and U_2 are row equivalent to $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$, but not U_3

Problem 4.109

Form the matrices A, B, C corresponding to U_1, U_2 and U_3 and row reduce these matrices to canonical form

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 1 & 5 \\ 1 & 2 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 1 & 5 \\ 1 & 2 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & -1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 10 \\ 2 & 4 & 3 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 10 \\ 0 & 0 & -3 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

U_1 and U_3 are row equivalent to $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$, but not U_2.

Problem 4.110

- Row reduce the original matrix, M, to echelon form, we'll call this new matrix E.
- The rows of E containing pivots are a basis for the row space of M.
- The columns of M corresponding to columns of E with pivots are a basis for the column space of M.

4.110a

$$\begin{bmatrix} 0 & 0 & 3 & 1 & 4 \\ 1 & 3 & 1 & 2 & 1 \\ 3 & 9 & 4 & 5 & 2 \\ 4 & 12 & 8 & 8 & 7 \end{bmatrix} \xrightarrow[swap\{(1,3),(2,1)\}]{} \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 3 & 9 & 4 & 5 & 2 \\ 0 & 0 & 3 & 1 & 4 \\ 4 & 12 & 8 & 8 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 4 \\ 0 & 0 & -4 & 0 & -3 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

i) basis for the row space is $\{(1,3,1,2,1),(0,0,1,-1,-1),(0,0,0,4,7)\}$

ii) basis for the column space is C_1, C_3 and C_4 of the original matrix:

$\{(0,1,3,4)^T,(3,1,4,8)^T,(1,2,5,8)^T\}$

NOTE: the answer in the book for part ii) is incorrect as C_1 and C_2 are not linearly independent. Clearly $C_2 = 3C_1$.

4.110b

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 3 \\ 3 & 6 & 5 & 2 & 7 \\ 2 & 4 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -2 \\ 0 & 0 & -2 & -2 & -4 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

i) basis for the row space is $\{(1,2,1,0,1),(0,0,1,1,2)\}$

ii) basis of the column space is C_1, C_3 of the original matrix: $\{(1,1,3,2)^T,(1,2,5,1)^T\}$

Problem 4.111

A matrix is in a echelon form if the following two condition hold:

(1) All zero rows, if any, are at the bottom of the matrix.

(2) Each leading nonzero entry in a row is to the right of the leading nonzero entry in the preceding row.

If the deleted row is an all zero row(at the bottom of the matrix) it is clear that it can't interfere with any of the two conditions.

If the deleted row is not an all zero row, condition 1 obviously holds. If the deleted row is R_k and the two adjacent rows are R_{k-1}, R_{k+1} and the three pivots are denoted as $a_{(k-1)j_{k-1}}, a_{kj_k}, a_{(k+1)j_{k+1}}$ according to (2):

$(I) j_{k-1} < j_k < j_{k+1}$

(Note pivot of the kth row is located in the column number j_k).

In the resulting matrix R_{k-1} and R_{k+1} are located consequently and the first nonzero entries are $a_{(k-1)j_{k-1}}, a_{(k+1)j_{k+1}}$. From (I) we conclude that $a_{(k+1)j_{k+1}}$ is located on the right side of $a_{(k-1)j_{k-1}}$ then second condition holds. Therefore, the new matrix is in the echelon form.

Problem 4.113

(a)

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{rank(A)=2, rank(B)=2.}$$

$$A + B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \text{rank(A+B)=1} < \text{rank(A), rank(B).}$$

(b)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{rank(A)=1, rank(B)=1.}$$

$$A + B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \text{rank(A+B)=1} = \text{rank(A)} = \text{rank(B).}$$

(b)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{rank(A)=1, rank(B)=1.}$$

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{rank(A+B)=2} > \text{rank(A), rank(B).}$$

Problem 4.128

For each part write the given vector v as a linear combination of the two basis vectors;

$v = x(1,-2) + y(4,-7) = (x+4y, -2x-7y)$

4.128a

$v = (5,3)$

$(5,3) = (x+4y, -2x-7y)$

$5 = x + 4y$

$3 = -2x - 7y$

$$\begin{bmatrix} 1 & 4 \\ -2 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 4 & 5 \\ -2 & -7 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 4 & 5 \\ 0 & 1 & 13 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -1 & 0 & 47 \\ 0 & 1 & 13 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -47 \\ 0 & 1 & 13 \end{array}\right]$$

The solution is $x = -47, y = 13$; therefore, the coordinate vector of $(5,3)$ with respect to the given basis is $[-47, 13]$

NOTE: the answer in the text is incorrect.

4.128b

Do the same with the vector (a,b):

$(a,b) = (x+4y, -2x-7y)$

$a = x + 4y$

$b = -2x - 7y$

$$\begin{bmatrix} 1 & 4 \\ -2 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 4 & a \\ -2 & -7 & b \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 4 & a \\ 0 & 1 & 2a+b \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -1 & 0 & 8a+4b-a \\ 0 & 1 & 2a+b \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -7a-4b \\ 0 & 1 & 2a+b \end{array}\right]$$

Then $x = -7a - 4b, y = 2a + b$, and the coordinate vector of (a,b) with respect to the given basis is $[-7a - 4b, 2a + b]$.

Problem 4.129

$u_1 = (1, 2, 0), u_2 = (1, 3, 2), u_3 = (0, 1, 3)$

Let $v = wu_1 + xu_2 + yu_3 = w(1, 2, 0) + x(1, 3, 2) + y(0, 1, 3)$

$v = (w + x, 2w + 3x + y, 2x + 3y)$.

Then solve for w, x, y

4.129 a

$v = (2, 7, -4)$

$(2, 7, -4) = (w + x, 2w + 3x + y, 2x + 3y)$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 2 & 3 & 1 & 7 \\ 0 & 2 & 3 & -4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & -4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & 10 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & -10 \end{array}\right]$$

The coordinate vector of v with respect to the given basis is $[-11, 13, -10]$

4.129 b

$v = (a, b, c)$

$(a, b, c) = (w + x, 2w + 3x + y, 2x + 3y)$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 2 & 3 & 1 & b \\ 0 & 2 & 3 & c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b - 2a \\ 0 & 2 & 3 & c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b - 2a \\ 0 & 0 & 1 & c - 2(b - 2a) \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 0 & b-2a-(4a-2b+c) \\ 0 & 0 & 1 & 4a-2b+c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7a-3b+c \\ 0 & 1 & 0 & -6a+3b-c \\ 0 & 0 & 1 & 4a-2b+c \end{array}\right]$$

The coordinate vector of v with respect to the given basis is

$[7a-3b+c, -6a+3b-c, 4a-2b+c]$

Problem 4.130

4.130 a

We write the given polynomial as a linear combination of the three basis polynomials given:

$2t^3+t^2-4t+2 = w(t^3+t^2)+x(t^2+t)+y(t+1)+z(1)$

$2t^3+t^2-4t+2 = wt^3+(w+x)t^2+(x+y)t+(y+z)$

$w = 2$

$w+x = 1 \Rightarrow x = -1$

$x+y = -4 \Rightarrow y = -3$

$y+z = 2 \Rightarrow z = 5$

therefore $[v] = [2, -1, -3, 5]$

NOTE: the answer in the book is incorrect for this question.

4.130 b

$at^3+bt^2+ct+d = w(t^3+t^2)+x(t^2+t)+y(t+1)+z(1)$

$at^3+bt^2+ct+d = wt^3+(w+x)t^2+(x+y)t+(y+z)$

$w = a$

$w+x = b \Rightarrow x = b-a$

$x+y = c \Rightarrow y = c-b+a$

$y+z = d \Rightarrow z = d-c+b-a$

therefore $[v] = [a, -a+b, a-b+c, -a+b-c+d]$

NOTE: The text answer is incorrect for this problem.

4.131

(a)

$$x \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + y \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + z \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 6 & 7 \end{bmatrix}$$

$$\begin{cases} x+y+z & = 3 \\ x-y & = -5 \\ x+y & = 6 \\ x+y & = 7 \end{cases}$$

This set of equations is not solvable since the last two equations are in contradiction. Therefore the requested vector is not in the span off the other three vectors.

(b)

$$x \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + y \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + z \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{cases} x+y+z & = a \\ x-y & = b \\ x+y & = c \\ x+y & = d \end{cases}$$

This set of equations might have an answer assuming c=d. To solve them we find the row-canonical form of the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 1 & -1 & 0 & b \\ 1 & 1 & 0 & c \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & -2 & -1 & b-a \\ 0 & 0 & -1 & c-a \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & \frac{1}{2} & \frac{a-b}{2} \\ 0 & 0 & 1 & a-c \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{a+b}{2} \\ 0 & 1 & \frac{1}{2} & \frac{a-b}{2} \\ 0 & 0 & 1 & a-c \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{b+c}{2} \\ 0 & 1 & 0 & \frac{c-b}{2} \\ 0 & 0 & 1 & a-c \end{bmatrix} \to x = \frac{b+c}{2},\ y = \frac{c-b}{2},\ z = a-c$$

Problem 4.132

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 2 & 5 & -4 & 7 \\ 1 & 4 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The dimension is 2 (number of non-zero rows) and $\{t^3+2t^2-3t+4, t^2+2t-1\}$ is a basis.

Problem 4.133

Write the 3×6 matrices as vectors in R^6 (the stretched out form, or, formally, by using aprpopriate coordinates). Then $\begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 2 \\ 2 & 4 & 3 & 7 & 5 & 6 \\ 1 & 2 & 3 & 5 & 7 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 2 & 2 & 6 & 4 \end{bmatrix} \to$

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$U = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 2 \end{bmatrix}$. $\{U, V\}$ is a basis and the dimension is 2.

Problem 4.134

- a) FALSE. This would only be a basis if the three spanning vectors are ALSO linearly independent, which need not be true in general. e.g. $(1,1),(1,2),(2,1)$ span R^2
- b) TRUE. The maximum rank of a 4x8 matrix is 4 [the maximum possible number of independent rows = the total number of rows!]. This is also the rank of the transpose matrix, therefore only a maximum of four rows of the transpose can be linearly independent. In particular, any six of the rows of the transpose, i.e. six of the columns of the original matrix, MUST be linearly dependent.
- c) First clarify the meaning of the question. Let $W = span u1, u2, u3$. If we assume the question means the vector $w \in W$, then the answer is TRUE because the three given vectors would be a basis for W because they are linearly independent. However, if we assume W is intended to be a subspace of the larger space V and $w \in V$, $w \notin W$, then the four vectors would be linearly independent so the answer would be FALSE. As stated this question is ambiguous.
- d) Any basis for a vector space with four linearly independent vectors must have at least four vectors in it, therefore the dim is at least 4. The statement is TRUE.
- e) TRUE. Adding any vector from $\mathrm{Sp}\{u_1, u_2, u_3\}$ to the spanning set $\{u_1, u_2, u_3\}$ will not alter the span because it is already a linear combination of the original three vectors.
- f) TRUE. Any subset of a linearly independent set must be linearly independent.

Problem 4.135

- a) FALSE. Deleting any column does not alter the echelon structure. If the column contains a pivot p one of the following will become the new pivot instead of p [whichever comes first to the right of p]: (i) the first non-zero entry to the right of p and in the row containing p OR (ii) the next pivot entry of the original matrix. If the deleted column does not contain a pivot, then the pivots of the new matrix will be the same as those of the original matrix. So why false? Because the deleted column may contain a pivot which is in a row with NO other non-zero entries, in which case the new matrix have a zero row. This zero row may have other non-zero rows below it, therefore the matrix is not in echelon form.

- b) FALSE. [NB row canonical form is echelon form with: all pivots 1, and all entries above pivots zero.] The example in the text is good. Deleting the second column makes the entry 2 into a pivot entry, and there is a non-zero value above it. As well, in this example, the new pivot entry (2) is not a 1 as required for row-canonical form.

- c) TRUE. Deleting a non-pivot containing column does not alter the echelon structure of the matrix (as in part a) and neither does it alter the row-canonical nature, because the pivots are not altered. As in part a), deleting a column without a pivot cannot introduce a zero row, because every row must still contain at least one pivot.

Problem 4.136

In each case we derive a basis and the we count the elements of the basis to find the dimension. All matrices are $n \times n$.

(a)

$$S_1 = \left\{ A_k[a_{ij}], \begin{array}{l} a_{ij} = 0 \;\; if \;\; i \;\; or \;\; j \; \neq k \\ a_{ij} = 1 \;\; if \;\; i = j = k \end{array} |k = 1, 2, .., n \right\}$$

$$S_2 = \left\{ A_k[a_{ij}], \begin{array}{l} a_{ij} = 1 \; if \; (i,j) = (k,l) or (l,k), i \neq j \\ a_{ij} = 0 \;\; elsewhere \end{array} |k = 2, 3, .., n, l = 1, .., k-1 \right\}$$

$S_1 \bigcup S_2$ is basis for the symmetric matrices. $N(S_1) = n$ and $N(S_2) = (n-1)+(n-2)+...+1$. $N(S_1)+N(S_2) = n+n-1+...+1 = \frac{1}{2}n(n+1)$.

(b)

$$S_1 = \left\{ A_k[a_{ij}], \begin{array}{l} a_{ij} = 1 \; if \; (i,j) = (l,k), i \neq j \\ a_{ij} = -1 \; if \; (i,j) = (k,l), i \neq j \\ a_{ij} = 0 \;\; i = j \end{array} |k = 2, 3, .., n, l = 1, .., k-1 \right\}$$

S_1 is a basis for antisymmetric matrices and $N(S_1) = n-1+n-2+...+1 = \frac{1}{2}n(n-1)$

(c)

$$S_1 = \left\{ A_k[a_{ij}], \begin{array}{l} a_{ij} = 1 \; if \; i = j = k \\ a_{ij} = 0 \;\; elsewhere \end{array} |k = 2, 3, .., n \right\}$$

S_1 is a basis for diagonal matrices $N(S_1) = n$

(d)

A basis for the set of scalar matrices has only one member, therefore dimension of scalar matrices is 1.